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Wright functions as scale-invariant solutions of the diffusion-wave equation

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Abstract

The time-fractional diffusion-wave equation is obtained from the classical diffusion or wave equation by replacing the first- or second-order time derivative by a fractional derivative of order α $(0 < \alpha \leqslant 2)$. Using the similarity method and the method of the Laplace transform, it is shown that the scale-invariant solutions of the mixed problem of signalling type for the time-fractional diffusion-wave equation are given in terms of the Wright function in the case $0 < \alpha < 1$ and in terms of the generalized Wright function in the case $1 < \alpha < 2$. The reduced equation for the scale-invariant solutions is given in terms of the Caputo-type modification of the Erdélyi–Kober fractional differential operator.

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1. Introduction

We consider the one-dimensional time-fractional diffusion-wave equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \mathscr{D}\frac{\partial^2 u(x,t)}{\partial x^2}, \quad \mathscr{D} > 0, \; 0 < \alpha \leqslant 2, \tag{1}$$

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where $u = u(x,t)$ is assumed to be a causal function of time, i.e., vanishing for $t < 0$, and the fractional derivative is taken in the Caputo sense ([2–5,12,18]):

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n \in \mathbb{N}, \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,\tau)}{\partial \tau^n} \, d\tau, & n-1 < \alpha < n. \end{cases} \tag{2}$$

Following the terminology, introduced by Mainardi [22], we refer to Eq. (1) as to the *fractional diffusion* and to the *fractional wave* equation in the cases $0 < \alpha \leqslant 1$ and $1 < \alpha \leqslant 2$, respectively. The difference between these two cases can be seen in the formula for the Laplace transform of the Caputo fractional derivative of order α ($n-1 < \alpha \leqslant n$, $n \in \mathbb{N}$):

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} \div s^\alpha \tilde{u}(x,s) - \sum_{k=0}^{n-1} s^{\alpha-1-k} \frac{\partial^k u(x,t)}{\partial t^k} \Big|_{t=0+} \tag{3}$$

with $\div$ denoting the juxtaposition of a function $\varphi(t)$ with its Laplace transform $\tilde{\varphi}(s)$. As a consequence, the initial and boundary conditions for the mixed boundary-value problem of signaling type for Eq. (1) have the form

$$u(x,0+) = u_0(x), \quad x > 0, \tag{4}$$

$$u(0+,t) = v_0(t), \quad u(+\infty,t) = v_1(t), \quad t > 0 \tag{5}$$

in the case $0 < \alpha \leqslant 1$ and

$$u(x,0+) = u_0(x), \quad \dot{u}(x,0+) = u_1(x), \quad x > 0, \tag{6}$$

$$u(0+,t) = v_0(t), \quad u(+\infty,t) = v_1(t), \quad t > 0 \tag{7}$$

in the case $1 < \alpha \leqslant 2$ (with abbreviation $\dot{u} = \partial u/\partial t$). As usual for the mixed boundary-value problems, a compatibility condition should be also added:

$$u_0(+\infty) = v_1(0). \tag{8}$$

Mathematical aspects of the boundary-value problems for Eq. (1) (with the fractional derivative in Caputo, Riemann–Liouville, inverse Riesz potential, etc., sense) and for other equations of this type have been treated in papers by several authors including Engler [7], Fujita [8], Gorenflo et al. [11], Gorenflo and Mainardi [13], Mainardi [19–22], Prüss [34], Saichev and Zaslavsky [35], Samko et al. [36], Schneider and Wyss [37], Wyss [44].

From the other side, some partial differential equations of fractional order of type (1) were successfully used for modelling relevant physical processes (see, for example [4,10,15,20,25,27–29,32,33] and references there). In applications, special types of solutions, which are invariant under some subgroup of the full symmetry group of the given equation (or for a system of equations) are especially important.

Recently, the scale-invariant solutions for Eq. (1) (with the fractional derivative in the Riemann–Liouville sense) and for the more general time- and space-fractional partial differential equation (with the Riemann–Liouville space-fractional derivative of order $\beta \leqslant 2$ instead of the second-order space derivative in Eq. (1)) have been presented by Buckwar and Luchko [1] and Luchko and

Gorenflo [17], respectively. To obtain these solutions, the groups of scaling transformations were first evaluated and then used for deriving the corresponding equations for the scale-invariant solutions of the initial problems. These equations are ordinary differential equations of fractional order with the new independent variables ($y = xt^{-\alpha/2}$ and $xt^{-\alpha/\beta}$, respectively). The derivatives are the Erdélyi–Kober derivatives (left- and right-hand sided) depending on the parameters α, β of the equations and on a parameter γ of the group of scaling transformations. To get an exact form of solutions of these equations, the corresponding operational calculi for the compositions of the left- and right-hand sided Erdélyi–Kober differential and integral operators have been used.

In this article we shall use another method to determine the scale-invariant solutions of the signalling problems $\{(1), (4), (5), (8)\}$ and $\{(1), (6), (7), (8)\}$. Using the similarity method we first find the group of scaling transformations for Eq. (1) and then determine the form of the boundary and initial conditions which are invariant under this group. Thus obtained signalling problems for Eq. (1) are solved by using the Laplace transform method in terms of the Wright and the generalized Wright functions. Specializing the equation of fractional order with the Caputo-type modification of the Erdélyi–Kober derivative, obtained for determining the scale-invariant solutions of (1), we get an integro-differential equation for the auxiliary function $F(y, \alpha/2)$, $y = xt^{-\alpha/2}$ introduced by Mainardi in connection with his construction of the Green function for the signalling problem for (1). It turns out that the Green function $F(y, \alpha/2)$ as well as the Green function $M(y, \alpha/2)$, $y = xt^{-\alpha/2}$ for the Cauchy problem for Eq. (1) (see [22,24]) are particular cases of the Wright function. Because of the important role of Wright functions in the theory of partial differential equations of fractional order of type (1) we give some elements of their theory.

2. The Wright and the generalized Wright functions

The entire function

$$W_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\rho k + \mu)}, \quad z \in \mathbb{C} \tag{9}$$

was introduced for the first time in the case $\rho > 0$ by Wright in his paper [40] in connection with his investigations in the asymptotic theory of partitions. In this paper and in the paper [41] some elementary properties and the asymptotics of function (9) were given. To get the asymptotic expansion of the function $W_{\rho,\mu}(z)$ Wright used the saddle points method and the integral representation

$$W_{\rho,\mu}(z) = \frac{1}{2\pi i} \int_{\mathrm{Ha}} \exp\{u + zu^{-\rho}\} u^{-\mu} \, du, \quad \rho > -1, \tag{10}$$

where Ha denotes the Hankel path in the u-plane with the cut along the line $\arg u = \pi$ starting from $-\infty$ on the real axis, passing round the origin in a counterclockwise direction and returning to $-\infty$, thus enclosing the cut. We give here this important result of Wright.

Proposition 1. *If* $\arg(-z) = \zeta$, $|\zeta| \leqslant \pi$, and

$$Z_1 = (\rho|z|)^{1/(\rho+1)} e^{i(\zeta+\pi)/(\rho+1)}, \quad Z_2 = (\rho|z|)^{1/(\rho+1)} e^{i(\zeta-\pi)/(\rho+1)},$$

then we have

$$W_{\rho,\mu}(z) = H(Z_1) + H(Z_2), \tag{11}$$

where

$$H(Z) = Z^{1/2-\mu} \mathrm{e}^{\{1+(1/\rho)\}Z} \left\{ \sum_{m=0}^{M} \frac{(-1)^m a_m}{Z^m} + \mathrm{O}\left(\frac{1}{|Z|^{M+1}} \right) \right\}, \quad Z \to \infty.$$

The numbers a_m can be exactly calculated for a given values of m, for example, $a_0 = (2\pi(\rho+1))^{-1/2}$. Due to the relation

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu W_{1,\nu+1}\left(-\frac{1}{4}z^2\right), \tag{12}$$

Wright considered the function $W_{\rho,\mu}(z)$ as a generalization of the Bessel function $J_\nu(z)$. Pathak [31] has found that the Wright function in the case of rational $\rho = p/q$ can be represented in terms of the Meijer G-function:

$$W_{\rho,\mu}(-z) = (2\pi)^{(p-q)/2} q^{1/2} p^{-\mu+1/2} \times G^{q,0}_{0,p+q}\left[\frac{z^q}{q^q p^p} \,\middle|\, \begin{matrix} - \\ 0, \frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}, 1-\frac{\mu}{p}, 1-\frac{1+\mu}{p}, \ldots, 1-\frac{p-1+\mu}{p} \end{matrix} \right]. \tag{13}$$

Combining this formula with the lists of particular cases of the G-function (see, i.e., [16,26]) one can obtain many representations of the function $W_{\rho,\mu}(z)$ in the case of rational $\rho = p/q$ in terms of elementary and special functions of hypergeometric type. In the general case of arbitrary positive real ρ the Wright function is a particular case of the Fox H-function ([14,16, App. E; [39]]; [Chapter 1]):

$$W_{\rho,\mu}(-z) = H^{1,0}_{0,2}\left[z \,\middle|\, \begin{matrix} - \\ (0,1), (1-\mu, \rho) \end{matrix} \right]. \tag{14}$$

$W_{\rho,\mu}(z)$ is still an entire function if $-1 < \rho < 0$, but its asymptotic behaviour presents certain new features in comparison with the case $\rho > 0$. It was shown by Wright [43] that for $z \to \infty$ the function $W_{\rho,\mu}(z)$ is exponentially small in a suitable sector containing the negative real semi-axis, exponentially large in two neighbouring sectors and, if $-1 < \rho < -\frac{1}{3}$, it has an algebraic expansion in a sector containing the positive real semi-axis.

The important particular cases of the Wright function, namely, the functions $M(z;\beta) = W_{-\beta,1-\beta}(-z)$ and $F(z;\beta) = W_{\beta,0}(-z)$ in the case $0 < \beta < 1$ have been considered in details in [22,24]. For $\beta = 1/q$ with an integer $q \geqslant 2$, these functions can be expressed as a sum of $(q-1)$ simpler entire functions. In the simplest cases $q = 2$ and 3 it was shown that

$$M(z, \tfrac{1}{2}) = \frac{1}{\sqrt{\pi}} \exp(-z^2/4), \; M\left(z; \frac{1}{3}\right) = 3^{2/3} \mathrm{Ai}\left(\frac{z}{3^{1/3}}\right) \tag{15}$$

with the Airy function $\mathrm{Ai}(z)$. We recall also the Laplace transform pairs presented in [22] (see also [6, Chapter 1.3, 9,38]):

$$M(t;\beta) \div E_\beta(-s), \quad 0 < \beta < 1, \tag{16}$$

$$F(t^{-\beta};\beta)/t \div \exp(-s^\beta), \quad 0 < \beta < 1, \tag{17}$$

$$W_{-\rho,\mu}(-t) \div E_{\rho,\mu+\rho}(-s), \quad 0 < \rho < 1. \tag{18}$$

Here

$$E_\rho(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + 1)}, \quad \rho > 0, \ z \in \mathbb{C} \tag{19}$$

is the Mittag–Leffler function and

$$E_{\rho,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + \mu)}, \quad \rho > 0, \ z \in \mathbb{C} \tag{20}$$

is the generalized Mittag–Leffler function, respectively. For the theory of the Mittag–Leffler-type functions with special emphasis in theirs applications in fractional calculus we refer to the paper by Mainardi and Gorenflo [23].

We shall use also some properties of the generalized Wright function

$$W_{(\mu,a),(\nu,b)}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(a + \mu k)\Gamma(b + \nu k)}, \quad \mu, \nu \in \mathbb{R}, \ a, b \in \mathbb{C}. \tag{21}$$

Wright himself has investigated this function in the case $\mu > 0$, $\nu > 0$ in [42]. If $a=\mu=1$ or $b=\nu=1$, respectively, then it is reduced to the Wright function (9). In our further discussions we use the generalized Wright function with $0 < -\mu < \nu \leqslant 2$, a case considered by Luchko and Gorenflo [17]. We give here some properties of this function proved in [17].

Proposition 2. *The generalized Wright function* $W_{(\mu,a),(\nu,b)}(z)$ *is an entire function for* $0 < \mu + \nu$, $a, b \in \mathbb{C}$.

Proposition 3. *Let* $\gamma(\epsilon; \varphi)$ $(0 < \varepsilon, \ 0 < \varphi \leqslant \pi)$ *be a contour in the complex* ζ*-plane with nondecreasing* $\arg \zeta$ *consisting of the following parts*:

(i) *the ray* $\arg \zeta = -\varphi$, $\epsilon \leqslant |\zeta|$;

(ii) *the arc* $-\varphi \leqslant \arg \zeta \leqslant \varphi$ *of the circumference* $|\zeta| = \epsilon$;

(iii) *the ray* $\arg \zeta = \varphi$, $\epsilon \leqslant |\zeta|$.

Then, the generalized Wright function $W_{(\mu,a),(\nu,b)}(z)$ *has the representation*

$$W_{(\mu,a),(\nu,b)}(z) = \frac{1}{2\pi i} \int_{\gamma(\epsilon;\varphi)} e^{\zeta} \zeta^{-a} E_{\nu,b}(z\zeta^{-\mu}) \, d\zeta, \quad 0 < \varepsilon, \ \pi/2 < \varphi \leqslant \pi, \ 0 < \nu + \mu, \ 0 < \nu \tag{22}$$

with the generalized Mittag–Leffler function (20) *in the kernel.*

In the case $0 < \nu + \mu$, $0 < \mu$ one can get a similar representation but with $\zeta^{-b} E_{\mu,a}(z\zeta^{-\nu})$ instead of $\zeta^{-a} E_{\nu,b}(z\zeta^{-\mu})$ by using (22) and the symmetry of the indices in (21).

Like the Wright function in the case $-1 < \rho < -\frac{1}{3}$, the generalized Wright function has an algebraic asymptotic expansion on the positive real semi-axis, if the parameters are suitably restricted.

Proposition 4. *Let* $0 < \nu/3 < -\mu < \nu \leqslant 2$, $L, P \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. *Then,*

$$W_{(\mu,a),(\nu,b)}(x) = \sum_{l=0}^{L-1} \frac{x^{(a-1-l)/(-\mu)}}{(-\mu)\Gamma(l+1)\Gamma(b + \nu(a - l - 1)/(-\mu))}$$

$$-\sum_{k=1}^{P}\frac{x^{-k}}{\Gamma(b-\nu k)\Gamma(a-\mu k)}+\mathrm{O}(x^{(a-1-L)/(-\mu)})+\mathrm{O}(x^{-1-P}),\quad x\to+\infty. \tag{23}$$

3. Invariants of scaling transformations for Eq. (1)

Since the scale-invariant solutions for the diffusion equation ($\alpha=1$ in (1)) and the wave equation ($\alpha=2$ in (1)) are well known (see, for example, [30]) we restrict ourselves in the further discussions to the case $0<\alpha<2$, $\alpha\neq 1$.

Let G be a one parameter group of scaling transformations for Eq. (1) of the form $G\circ(x,t,u)=(\lambda x,\ \lambda^b t,\ \lambda^c u)$. It implies that if $u=f(x,t)$ is a solution of Eq. (1), so is the function $u_\lambda=\lambda^c f(\lambda^{-1}x,\lambda^{-b}t)$, where λ is any positive real number. We then have ($\bar{x}=\lambda^{-1}x$, $\bar{t}=\lambda^{-b}t$):

$$\frac{\partial^2 u_\lambda}{\partial x^2}=\lambda^{c-2}\frac{\partial^2 f(\bar{x},\bar{t})}{\partial \bar{x}^2}$$

and for $n-1<\alpha<n$, $n\in\mathbb{N}$

$$\begin{aligned}\frac{\partial^\alpha u_\lambda}{\partial t^\alpha}&=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-\tau)^{n-\alpha-1}\frac{\partial^n u_\lambda}{\partial\tau^n}\,\mathrm{d}\tau\\&=\frac{\lambda^{c-\alpha b}}{\Gamma(n-\alpha)}\int_0^{\bar{t}}(\bar{t}-\bar{\tau})^{n-\alpha-1}\frac{\partial^n f(\bar{x},\bar{\tau})}{\partial\bar{\tau}^n}\,\mathrm{d}\bar{\tau}=\lambda^{c-\alpha b}\frac{\partial^\alpha f(\bar{x},\bar{t})}{\partial\bar{t}^\alpha}.\end{aligned}$$

Using the obtained relations, we get

$$\frac{\partial^\alpha u_\lambda}{\partial t^\alpha}-\mathscr{D}\frac{\partial^2 u_\lambda}{\partial x^2}\equiv\lambda^{c-\alpha b}\frac{\partial^\alpha f(\bar{x},\bar{t})}{\partial\bar{t}\alpha}-\mathscr{D}\lambda^{c-2}\frac{\partial^2 f(\bar{x},\bar{t})}{\partial\bar{x}^2},$$

which implies that if the function $u=f(x,t)$ is a solution of (1) then u_λ is also a solution of (1) for any $\lambda>0$ if and only if

$$\alpha b=2. \tag{24}$$

According to the general theory (see [30]) the invariants of the group G of scaling transformations $G\circ(x,t,u)=(\lambda x,\ \lambda^b t,\ \lambda^c u)$ are given by the expressions

$$\eta(x,t)=xt^{-1/b},\qquad \zeta(x,t,u)=t^{-c/b}u. \tag{25}$$

Specializing formulae (25) to the case of Eq. (1) we get the following result:

Theorem 5. *The invariants of the group G of scaling transformations $G\circ(x,t,u)=(\lambda x,\ \lambda^{2/\alpha}t,\ \lambda^c u)$ for Eq.* (1) *are given by the expressions*

$$y=xt^{-\alpha/2},\qquad v=t^{-\gamma}u \tag{26}$$

with a real parameter $\gamma=c\alpha/2$.

Since we are looking for the scale-invariant solutions of the signalling problems {(1), (4), (5), (8)} and {(1), (6), (7), (8)}, let us find the form of the initial and boundary conditions (4)–(5) and (6)–(7) which are invariant under the obtained group of scaling transformations for Eq. (1).

I. *The case* $0 < \alpha < 1$: Let $u = f(x,t)$ be a solution of the problem $\{(1), (4), (5), (8)\}$. The functions $u_\lambda = \lambda^{2\gamma/\alpha} f(\lambda^{-1}x, \lambda^{-2/\alpha}t)$, $\lambda > 0$, should also be solutions of this problem. We then have

$$u_\lambda(x,t)|_{t\to 0+} = u_0(x), \quad x > 0,$$

but also

$$u_\lambda(x,t)|_{t\to 0+} = \lambda^{2\gamma/\alpha} f(\lambda^{-1}x, \lambda^{-2/\alpha}t)|_{t\to 0+} = \lambda^{2\gamma/\alpha} u_0(\lambda^{-1}x).$$

These relations imply

$$\lambda^{2\gamma/\alpha} u_0(\lambda^{-1}x) = u_0(x).$$

Substituting here $\lambda = x$ we arrive at

$$u_0(x) = u_0(1)x^{2\gamma/\alpha} = C_1 x^{2\gamma/\alpha} \tag{27}$$

with an arbitrary constant C_1. Analogously, we have for the boundary conditions (5), with arbitrary constants C_3 and C_4,

$$v_0(t) = v_0(1)t^\gamma = C_3 t^\gamma, \quad v_1(t) = v_1(1)t^\gamma = C_4 t^\gamma. \tag{28}$$

Comparing the compatibility condition (8) with relations (27), (28), we get the two possibilities

$$\gamma \neq 0, \quad C_1 = C_4 = 0, \tag{29}$$

$$\gamma = 0, \quad u_0(x) = C_1 = C_4 = v_1(t), \quad x > 0,\ t > 0, \tag{30}$$

which together with relations (27) and (28) give us the form of the initial and boundary conditions of the signalling problem $\{(1), (4), (5), (8)\}$ which are invariant under the group of the scaling transformations of Eq. (1).

II. *The case* $1 < \alpha < 2$: In addition to relations (27), (28) and restrictions (29), (30), we deduce the other scale-invariant initial condition. We have

$$\lim_{t\to 0+} \frac{\partial u_\lambda}{\partial t} = u_1(x)$$

and ($\bar{t} = \lambda^{-2/\alpha}t$)

$$\lim_{t\to 0+} \frac{\partial u_\lambda}{\partial t} = \lim_{t\to 0+} \frac{\partial}{\partial t} \lambda^{2\gamma/\alpha} f(\lambda^{-1}x, \lambda^{-2/\alpha}t)$$
$$= \lambda^{2\gamma/\alpha - 2/\alpha} \lim_{\bar{t}\to 0+} \frac{\partial f(\lambda^{-1}x, \bar{t})}{\partial \bar{t}} = \lambda^{2\gamma/\alpha - 2/\alpha} u_1(\lambda^{-1}x).$$

It follows from the last two relations that

$$u_1(x) = \lambda^{2\gamma/\alpha - 2/\alpha} u_1(\lambda^{-1}x).$$

Substituting here the value $\lambda = x$ we have

$$u_1(x) = u_1(1)x^{(2/\alpha)(\gamma-1)} = C_2 x^{(2/\alpha)(\gamma-1)} \tag{31}$$

with an arbitrary constant C_2. Combining the last formula with relations (27), (28), restrictions (29), (30) we get the form of the initial and boundary conditions of the signalling problem $\{(1), (6), (7), (8)\}$ which are invariant under the group of the scaling transformations of Eq. (1).

4. Scale-invariant solutions

Using the Laplace transform method we find in this section the exact form of the scale-invariant solutions of the signalling problems {(1), (4), (5), (8)} and {(1), (6), (7), (8)}. As we have seen, such solutions exist only for the initial and boundary conditions of the form (27), (28), (31) with restrictions (29), (30). We consider now all these cases, combining relations (29) and (30) with cases I and II.

I_1. $0 < \alpha < 1$ and the initial and boundary conditions (4), (5) are given by the equations

$$u(x,0+) = u_0(x) = 0, \quad x > 0,$$

$$u(0+,t) = v_0(t) = C_3 t^{\gamma}, \quad u(+\infty,t) = v_1(t) = 0, \quad t > 0.$$

In this case we can use the solution formula of the signalling problem {(1), (4), (5), (8)} from Mainardi [22], namely

$$u(x,t) = \int_0^t \mathcal{G}_s(x, t-\tau; \alpha/2) v_0(\tau)\,\mathrm{d}\tau, \tag{32}$$

where the *Green function* $\mathcal{G}_s(x,t;\alpha/2)$ is given by

$$\mathcal{G}_s(x,t;\alpha/2) = F(y/\sqrt{\mathcal{D}};\alpha/2)/t. \tag{33}$$

Here $y = xt^{-\alpha/2}$ is the first scale-invariant (26) of Eq. (1) and the auxiliary function $F(z;\beta)$ is a particular case of the Wright function

$$F(z;\beta) = W_{-\beta,0}(-z). \tag{34}$$

Integral (32) with $v_0(\tau) = C_3\tau^{\gamma}$ is convergent if $\gamma > -1$. To get its value, we note that it is a Laplace convolution of the functions

$$f(t) = F\left(\frac{x}{\sqrt{\mathcal{D}}} t^{-\alpha/2}; -\alpha/2\right)/t \quad \text{and} \quad g(t) = C_3 t^{\gamma}.$$

Combining the Laplace transform pair (17) with

$$g(t) \div C_3 \Gamma(1+\gamma) s^{-1-\gamma},$$

we arrive at the Laplace transform of the solution $u(x,t)$:

$$u(x,t) \div C_3\Gamma(1+\gamma)s^{-1-\gamma}\mathrm{e}^{(-x/\sqrt{\mathcal{D}})s^{\alpha/2}}.$$

Since $\alpha/2 < 1$, we transform the integration contour in the complex Laplace inversion formula into the Hankel path Ha and get

$$u(x,t) = C_3\Gamma(1+\gamma)\frac{1}{2\pi \mathrm{i}}\int_{\mathrm{Ha}} \mathrm{e}^{st} s^{-1-\gamma} \mathrm{e}^{(-x/\sqrt{\mathcal{D}})s^{\alpha/2}}\,\mathrm{d}s.$$

Comparing this formula with (10) we obtain the scale invariant solution of the signalling problem {(1), (4), (5), (8)} in the form

$$u(x,t) = C_3\Gamma(1+\gamma)t^{\gamma} W_{-\alpha/2,1+\gamma}\left(\frac{-xt^{-\alpha/2}}{\sqrt{\mathcal{D}}}\right), \quad \gamma > -1.$$

I_2. $0 < \alpha < 1$, the initial and boundary conditions (4), (5) are given as

$$u(x,0+) = u_0(x) = C_1, \quad x > 0,$$

$$u(0+,t) = v_0(t) = C_3, \quad u(+\infty,t) = v_1(t) = C_1, \quad t > 0.$$

In this case, applying the Laplace transform to Eq. (1) and using formula (3), we get for the Laplace transform $\tilde{u}(x,s)$ of the solution of the problem under consideration the ordinary differential equation

$$s^\alpha \tilde{u}(x,s) = \mathscr{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\tilde{u}(x,s) + C_1 s^{\alpha-1} \tag{35}$$

with the boundary conditions

$$\tilde{u}(0+,s) = C_3/s, \;\; \tilde{u}(+\infty,s) = C_1/s. \tag{36}$$

The solution of problem $\{(35), (36)\}$ has the form

$$\tilde{u}(x,s) = \frac{C_3 - C_1}{s}\mathrm{e}^{(-x/\sqrt{\mathscr{D}})s^{\alpha/2}} + \frac{C_1}{s}.$$

Applying the complex Laplace inversion formula, transforming the integration contour into the Hankel path and using (10), we find

$$u(x,t) = (C_3 - C_1)W_{-\alpha/2,1}\left(\frac{-xt^{-\alpha/2}}{\sqrt{\mathscr{D}}}\right) + C_1.$$

We thus have proved the following result:

Theorem 6. *The scale-invariant solutions of signalling problem* $\{(1), (4), (5), (8)\}$ *for the fractional diffusion equation with respect to the group* G *of scaling transformations* $G \circ (x,t,u) = (\lambda x,\ \lambda^{2/\alpha}t,\ \lambda^{2\gamma/\alpha}u)$ *have the form*

$$u(x,t) = C_3\Gamma(1+\gamma)t^\gamma W_{-\alpha/2,1+\gamma}\left(\frac{-y}{\sqrt{\mathscr{D}}}\right) \tag{37}$$

in the case $-1 < \gamma,\ \gamma \neq 0$ *and*

$$u(x,t) = (C_3 - C_1)W_{-\alpha/2,1}\left(\frac{-y}{\sqrt{\mathscr{D}}}\right) + C_1 \tag{38}$$

in the case $\gamma = 0$, *where* $y = xt^{-\alpha/2}$ *is the first scale invariant* (26), $W_{\lambda,\mu}(z)$ *is the Wright function given by* (9) *and* C_1, C_3 *are arbitrary constants.*

The scale-invariant solutions of problem $\{(1), (6), (7), (8)\}$ in the case $1 < \alpha < 2$, that is, for the fractional wave equation, exist if and only if the initial and boundary conditions (6), (7) have the form II_1 or II_2:

II_1. $\gamma \neq 0$:

$$u(x,0+) = u_0(x) = 0, \quad \dot{u}(x,0+) = u_1(x) = C_2 x^{(2/\alpha)(\gamma-1)}, \quad x > 0,$$

$$u(0+,t) = v_0(t) = C_3 t^\gamma, \quad u(+\infty,t) = v_1(t) = 0, \quad t > 0,$$

II_2. $\gamma = 0$:

$$u(x,0+) = u_0(x) = C_1, \quad \dot{u}(x,0+) = u_1(x) = C_2 x^{-2/\alpha}, \quad x > 0,$$

$$u(0+,t) = v_0(t) = C_3, \quad u(+\infty,t) = v_1(t) = C_1, \quad t > 0.$$

We solve problem {(1), (6), (7), (8)} in the more general case, containing both cases II_1 and II_2:

$$u(x,0+) = u_0(x) = C, \quad \dot{u}(x,0+) = u_1(x) = Bx^{\beta}, \tag{39}$$

$$-2 < \beta < 0,\ \beta \neq -1,\ x > 0,$$

$$u(0+,t) = v_0(t), \quad u(+\infty,t) = v_1(t) = C, \quad t > 0. \tag{40}$$

Here C and B are arbitrary constants, and we suppose also for the sake of simplicity that there exists the Laplace transform $\tilde{v}_0(s)$ of the function $v_0(t)$.

We apply the Laplace transform to Eq. (1) and to the boundary conditions (40). Using formula (3) we get for the Laplace transform $\tilde{u}(x,s)$ of the solution the ordinary differential equation

$$s^{\alpha}\tilde{u}(x,s) = \mathscr{D}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\tilde{u}(x,s) + Cs^{\alpha-1} + Bx^{\beta}s^{\alpha-2} \tag{41}$$

with the boundary conditions

$$\tilde{u}(0+,s) = v_0(s), \quad \tilde{u}(+\infty,s) = C/s. \tag{42}$$

It can be directly checked that in the case $-2 < \beta < 0$, $\beta \neq -1$ the function

$$\tilde{u}_p(x,s) = \frac{C}{s} - \frac{B}{\mathscr{D}}\Gamma(1+\beta)x^{2+\beta}s^{\alpha-2}E_{2,3+\beta}(s^{\alpha}x^2/\mathscr{D}) \tag{43}$$

is a particular solution of Eq. (41). Here $E_{\rho,\mu}(z)$ is the generalized Mittag–Leffler function (20).

The general solution of Eq. (41) thus has the form

$$\tilde{u}(x,s) = C_1(s)\mathrm{e}^{-xs^{\alpha/2}/\sqrt{\mathscr{D}}} + C_2(s)\mathrm{e}^{xs^{\alpha/2}/\sqrt{\mathscr{D}}} + \tilde{u}_p(x,s) \tag{44}$$

with arbitrary functions $C_1(s)$, $C_2(s)$. Applying the asymptotic formula for the Mittag–Leffler function from [6, Chapter 1.3] we get

$$\tilde{u}_p(x,s) = \frac{C}{s} - \frac{B}{2}\Gamma(1+\beta)\mathscr{D}^{\beta/2}s^{-2-\alpha\beta/2}\mathrm{e}^{xs^{\alpha/2}/\sqrt{\mathscr{D}}} + \mathcal{O}(x^{\beta}), \quad x \to +\infty.$$

From the boundary conditions (42) and using the last formula we can find the form of the functions $C_1(s)$, $C_2(s)$ in (44):

$$C_1(s) = \tilde{v}_0(s) - \frac{C}{s} - \frac{B}{2}\Gamma(1+\beta)\mathscr{D}^{\beta/2}s^{-2-\alpha\beta/2},$$

$$C_2(s) = \frac{B}{2}\Gamma(1+\beta)\mathscr{D}^{\beta/2}s^{-2-\alpha\beta/2}.$$

We therefore arrive at the Laplace transform $\tilde{u}(x,s)$ of the solution in the form

$$\tilde{u}(x,s) = \tilde{u}_1(x,s) + \tilde{u}_2(x,s), \tag{45}$$

where

$$\tilde{u}_1(x,s) = \left(\tilde{v}_0(s) - \frac{C}{s} - \frac{B}{2}\Gamma(1+\beta)\mathscr{D}^{\beta/2}s^{-2-\alpha\beta/2}\right)\mathrm{e}^{-xs^{\alpha/2}\sqrt{\mathscr{D}}} + \frac{C}{s},$$

$$\tilde{u}_2(x,s) = \frac{B}{2}\Gamma(1+\beta)\mathscr{D}^{\beta/2}s^{-2-\alpha\beta/2}\mathrm{e}^{xs^{\alpha/2}/\sqrt{\mathscr{D}}} - \frac{B}{\mathscr{D}}\Gamma(1+\beta)x^{2+\beta}s^{\alpha-2}E_{2,3+\beta}(s^{\alpha}x^2/\mathscr{D}).$$

To obtain $u_1(x,t)$ from $\tilde{u}_1(x,s)$, we apply the Laplace convolution theorem and the same considerations as in cases I_1 and I_2 to get

$$u_1(x,t) = \int_0^t \mathscr{G}_s(x,t-\tau;\alpha/2)v_0(\tau)\,\mathrm{d}\tau \;- CW_{-\alpha/2,1}\left(-\frac{xt^{-\alpha/2}}{\sqrt{\mathscr{D}}}\right) - \frac{B}{2}\Gamma(1+\beta)\mathscr{D}^{\beta/2}t^{1+\alpha\beta/2}W_{-\alpha/2,2+\alpha\beta/2}\left(-\frac{xt^{-\alpha/2}}{\sqrt{\mathscr{D}}}\right) + C, \tag{46}$$

the Green function $\mathscr{G}_s(x,t;\alpha/2)$ being given by (33). Let us evaluate the inverse Laplace transform of the function $\tilde{u}_2(x,s)$. Using the asymptotic formula for the Mittag–Leffler function from [6, Chapter 1.3], we obtain

$$\tilde{u}_2(x,s) = \mathscr{O}(s^{-2}), \quad |s| \to \infty, \quad |\arg(s)| < \frac{2}{\alpha}\frac{\pi}{2}.$$

This formula and the fact that the function $\tilde{u}_2(x,s)$ is analytic in the complex s-plane with the cut along the negative real semi-axis allow us to transform the contour in the complex Laplace inversion formula to the contour $\gamma(\epsilon;\varphi)$ $(0<\varepsilon,\ \pi/2<\varphi<(2/\alpha)\pi/2)$. Here $\gamma(\epsilon;\varphi)$ $(0<\varepsilon,\ 0<\varphi\leqslant 2\pi)$ is the same contour as in the integral representation (22). We thus have

$$u_2(x,t) = \frac{1}{2\pi\mathrm{i}}\int_{\gamma(\epsilon;\varphi)} \mathrm{e}^{ts}\tilde{u}_2(x,s)\,\mathrm{d}s.$$

Using now (10) with the contour $\gamma(\epsilon;\varphi)$, $\pi/2<\varphi<(2/\alpha)\pi/2$ instead of the Hankel path Ha and representation (22) we get

$$u_2(x,t) = \frac{B}{2}\Gamma(1+\beta)\mathscr{D}^{\beta/2}t^{1+\alpha\beta/2}W_{-\alpha/2,2+\alpha\beta/2}\left(\frac{xt^{-\alpha/2}}{\sqrt{\mathscr{D}}}\right) - \frac{B}{\mathscr{D}}\Gamma(1+\beta)x^{2+\beta}t^{1-\alpha}W_{(-\alpha,2-\alpha),(2,3+\beta)}\left(\frac{x^2t^{-\alpha}}{\mathscr{D}}\right). \tag{47}$$

As a consequence of (45) we obtain the solution of problem {(1), (6), (7), (8)} with the initial and boundary conditions (39), (40) in the form

$$u(x,t) = u_1(x,t) + u_2(x,t), \tag{48}$$

where the functions $u_1(x,t)$ and $u_2(x,t)$ are given by (46) and (47).

Specializing formulae (47)–(48) to our cases II_1 and II_2 we get the following result:

Theorem 7. *The scale-invariant solutions of the signalling problem* {(1), (6), (7), (8)} *for the fractional wave equation with respect to the group* G *of scaling transformations* $G \circ (x,t,u) = (\lambda x,\ \lambda^{2/\alpha} t,\ \lambda^{2\gamma/\alpha} u)$ *have the form*

$$\begin{aligned} u(x,t) = {} & \left(C_3 - \frac{C_2 \mathscr{D}^{(\gamma-1)/\alpha}}{2} \Gamma(1+2(\gamma-1)/\alpha) \right) t^\gamma W_{-\alpha/2,1+\gamma}\left(\frac{-y}{\sqrt{\mathscr{D}}} \right) \\ & + C_2 \Gamma(1+2(\gamma-1)/\alpha) t^\gamma \left(\frac{\mathscr{D}^{(\gamma-1)/\alpha}}{2} W_{-\alpha/2,1+\gamma}\left(\frac{y}{\sqrt{\mathscr{D}}} \right) \right. \\ & \left. - \frac{y^{2+2(\gamma-1)/\alpha}}{\mathscr{D}} W_{(-\alpha,2-\alpha),(2,3+2(\gamma-1)/\alpha)}\left(\frac{y^2}{\mathscr{D}} \right) \right), \end{aligned} \tag{49}$$

in the case $1-\alpha < \gamma < 1,\ \gamma \neq 1-\alpha/2,\ \gamma \neq 0$ *and*

$$\begin{aligned} u(x,t) = {} & (C_3 - C_1 - C_2 \Gamma(1-2/\alpha) \mathscr{D}^{-1/\alpha}/2) W_{-\alpha/2,1}\left(\frac{-y}{\sqrt{\mathscr{D}}} \right) + C_1 \\ & + C_2 \Gamma(1-2/\alpha) \left(\frac{\mathscr{D}^{-1/\alpha}}{2} W_{-\alpha/2,1}\left(\frac{y}{\sqrt{\mathscr{D}}} \right) - \frac{y^{2-2/\alpha}}{\mathscr{D}} W_{(-\alpha,2-\alpha),(2,3-2/\alpha)}\left(\frac{y^2}{\mathscr{D}} \right) \right) \end{aligned} \tag{50}$$

in the case $\gamma = 0$. *In both cases* $y = xt^{-\alpha/2}$ *is the first scale invariant* (26), $W_{\lambda,\mu}(z)$ *is the Wright function* (9), $W_{(\mu,a),(\nu,b)}(z)$ *is the generalized Wright function* (21) *and* C_1, C_2, C_3 *are arbitrary constants.*

Remark 8. It is known (see for example [22]) that the Caputo fractional derivative (2) of order α coincides with the Riemann–Liouville fractional derivative

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n \in \mathbb{N}, \\ \dfrac{\partial^n}{\partial t^n} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} u(x,\tau)\, d\tau, & n-1 < \alpha < n \end{cases} \tag{51}$$

if and only if $\alpha \in \mathbb{N}$ or $\alpha \notin \mathbb{N}$ and $\partial^k u(x,t)/\partial t^k|_{t=0+} = 0,\ k = 0,\ldots,n-1 = [\alpha]$, both types of fractional derivative being assumed to exist. It means, that the scale-invariant solutions (37), (38), (49), and (50) of Eq. (1) with the Caputo fractional derivative will also be the scale-invariant solutions of the same equation with the Riemann–Liouville fractional derivative if and only if $C_1 = 0$ $(0 < \alpha < 1)$ or $C_1 = C_2 = 0$ $(1 < \alpha < 2)$. In fact, these solutions are the part of the scale-invariant solutions of Eq. (1) with the Riemann–Liouville fractional derivative obtained in Buckwar and Luchko [1] in the case $\gamma = 0$ and in Luchko and Gorenflo [17] in the general case.

5. Equation for the scale-invariant solutions

It follows from the general theory of Lie groups and Theorem 5 that the scale-invariant solutions of Eq. (1) should have the form

$$u(x,t) = t^\gamma v(y), \quad y = xt^{-\alpha/2}. \tag{52}$$

Furthermore, the general theory says that substituting (52) reduces the partial integro-differential equation (1) into an ordinary integro-differential equation with the unknown function $v(y)$. We shall now find this reduced equation.

As we have seen, the already found scale-invariant solutions (37), (38), (49), and (50) have the form (52). The corresponding parts of these solutions should also be solutions of the reduced equation. In particular, in this way we find an equation for the auxiliary function $F(y;\alpha/2)$ used for construction of the Green function (33) for the signalling problems $\{(1), (4), (5), (8)\}$ and $\{(1), (6), (7), (8)\}$.

Let us calculate the partial derivative u_{xx} and the partial fractional derivative $\partial^\alpha u/\partial t^\alpha$, $\alpha > 0$ in terms of derivatives of v. We find

$$u_x = t^{\gamma-\alpha/2}v'(y), \quad u_{xx} = t^{\gamma-\alpha}v''(y). \tag{53}$$

Using the relation ($z = x\tau^{-\alpha/2}$)

$$\tau\frac{\partial}{\partial\tau}\phi(z) = \tau x\left(-\frac{\alpha}{2}\right)\tau^{-\alpha/2-1}\phi'(z) = -\frac{\alpha}{2}z\frac{\mathrm{d}}{\mathrm{d}z}\phi(z), \tag{54}$$

we arrive for $n \in \mathbb{N}$ at

$$\begin{aligned}\frac{\partial^n u(x,\tau)}{\partial\tau^n} &= \frac{\partial^n}{\partial\tau^n}\tau^\gamma v(z) = \frac{\partial^{n-1}}{\partial\tau^{n-1}}\tau^{\gamma-1}\left(\gamma - \frac{\alpha}{2}z\frac{\mathrm{d}}{\mathrm{d}z}\right)v(z)\\ &= \cdots = \tau^{\gamma-n}\prod_{j=0}^{n-1}\left(\gamma - n + 1 + j - \frac{\alpha}{2}z\frac{\mathrm{d}}{\mathrm{d}z}\right)v(z).\end{aligned}$$

Applying this we get ($z = x\tau^{-\alpha/2}$, $n-1 < \alpha < n$, $n \in \mathbb{N}$)

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\frac{\partial^n}{\partial\tau^n}\tau^\gamma v(z)\,\mathrm{d}\tau\\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}\tau^{\gamma-n}\prod_{j=0}^{n-1}\left(\gamma - n + 1 + j - \frac{\alpha}{2}z\frac{\mathrm{d}}{\mathrm{d}z}\right)v(z)\,\mathrm{d}\tau.\end{aligned}$$

Now the substitution $\tau = t(y/u)^{2/\alpha}$ ($y = xt^{-\alpha/2}$ is the first scale invariant (26)) gives us

$$\begin{aligned}\frac{\partial^\alpha u}{\partial t^\alpha} &= \frac{2}{\alpha}\frac{t^{\gamma-\alpha}}{\Gamma(n-\alpha)}y^{2(\gamma-n+1)/\alpha}\int_y^\infty (u^{2/\alpha} - y^{2/\alpha})^{n-\alpha-1}u^{1-2\gamma/\alpha}\\ &\quad\times\prod_{j=0}^{n-1}\left(\gamma - n + 1 + j - \frac{\alpha}{2}u\frac{\mathrm{d}}{\mathrm{d}u}\right)v(u)\,\mathrm{d}u.\end{aligned} \tag{55}$$

Comparing this operator with the classical Erdélyi–Kober fractional differential operator of order α ($n-1 < \alpha \leqslant n \in \mathbb{N}$) [16, Chapter 1]

$$(P_\delta^{\tau,\alpha}g)(y) := \prod_{j=0}^{n-1}\left(\tau + j - \frac{1}{\delta}y\frac{\mathrm{d}}{\mathrm{d}y}\right)(K_\delta^{\tau+\alpha,n-\alpha}g)(y), \quad \delta > 0, \ \ y > 0, \tag{56}$$

where

$$(K_\delta^{\tau,\alpha}g)(y) := \begin{cases}\dfrac{\delta}{\Gamma(\alpha)}y^{\delta\tau}\displaystyle\int_y^\infty (u^\delta - y^\delta)^{\alpha-1}u^{-\delta(\tau+\alpha-1)-1}g(u)\,\mathrm{d}u, & \alpha > 0,\\ g(y), & \alpha = 0\end{cases} \tag{57}$$

is the Erdélyi–Kober fractional integral operator, we can consider the right-hand side of relation (55) as a particular case of the Caputo-type modification of the Erdélyi–Kober fractional differential operator of order α ($n-1<\alpha\leqslant n\in\mathbb{N}$). It can be defined in the general case ($\delta>0$, $\alpha>0$) by

$$({}_*P_\delta^{\tau,\alpha}g)(y):=\left(K_\delta^{\tau,n-\alpha}\prod_{j=0}^{n-1}\left(\tau+j-\frac{1}{\delta}u\frac{\mathrm{d}}{\mathrm{d}u}\right)g\right)(y),\quad y>0, \tag{58}$$

where the Erdélyi–Kober fractional integral operator $(K_\delta^{\tau,\alpha}g)(y)$ is given by (57). We then have ($n-1<\alpha\leqslant n\in\mathbb{N}$)

$$\frac{\partial^\alpha u}{\partial t^\alpha}=t^{\gamma-\alpha}({}_*P_{2/\alpha}^{\gamma-n+1,\alpha}v)(y). \tag{59}$$

Relations (53) and (59) allow us to establish the following result:

Theorem 9. *The reduced equation for the scale-invariant solutions of Eq.* (1) *of the form* (52) *is given by*

$$({}_*P_{2/\alpha}^{\gamma-n+1,\alpha}v)(y)=\mathscr{D}v''(y), \tag{60}$$

where the operator in the left-hand side is the Caputo-type modification of the Erdélyi–Kober fractional differential operator defined by (58).

Remark 10. As it follows from the definitions of the Caputo-type modification of the Erdélyi–Kober fractional differential operator (58) and the Erdélyi–Kober fractional integral operator (57) in the case $\alpha=n\in\mathbb{N}$, Eq. (60) for the scale-invariant solutions is a linear ordinary differential equation of order $\max\{n,2\}$. In the case $\alpha=1$ (the diffusion equation) we have

$$({}_*P_2^{\gamma,1}v)(y)=\left(\gamma-\frac{1}{2}y\frac{\mathrm{d}}{\mathrm{d}y}\right)v(y)$$

and (60) takes the form

$$\mathscr{D}v''(z)+\tfrac{1}{2}yv'(y)-\gamma v(y)=0. \tag{61}$$

In the case $\alpha=2$ (the wave equation) we get

$$\begin{aligned}({}_*P_1^{\gamma-1,2}v)(y)&=\left(\gamma-1-y\frac{\mathrm{d}}{\mathrm{d}y}\right)\left(\gamma-y\frac{\mathrm{d}}{\mathrm{d}y}\right)v(y)\\&=y^2v''(y)-2(\gamma-1)yv'(y)+\gamma(\gamma-1)v(y)\end{aligned}$$

and (60) is reduced to the ordinary differential equation of the second order:

$$(y^2-\mathscr{D})v''(y)-2(\gamma-1)yv'(y)+\gamma(\gamma-1)v(y)=0. \tag{62}$$

The complete discussion of these cases one can find, for example, in [30]. The case $\alpha=n\in\mathbb{N}$, $n>2$ was considered in [1].

Theorems 6 and 7 supply us with the solutions of the integro-differential equation (60) in the general case $0 < \alpha < 2$. In particular, in the cases $\gamma = 0$, $C_1 = 0$, $C_3 = 1$ $(0 < \alpha < 1)$ and $\gamma = 0$, $C_1 = C_2 = 0$, $C_3 = 1$ $(1 < \alpha < 2)$ the scale invariant solution of Eq. (1) has the form

$$v(y) = W_{-\alpha/2,1}(-y/\sqrt{\mathscr{D}}) = \int_0^t \mathscr{G}_s(x, t-\tau; \alpha/2)\,\mathrm{d}\tau, \tag{63}$$

where the Green function is given by (33) in terms of the auxiliary function $F(y;\alpha/2)$, $y = xt^{-\alpha/2}$. Differentiation of (63) with respect to t gives

$$-\frac{\alpha}{2} y v'(y) = F(y;\alpha/2). \tag{64}$$

Since $v(y)$ is a solution of Eq. (60), we arrive in the case $0 < \alpha < 1$, after inserting $v'(y)$ from (64) into (60), at the following equation for the auxiliary function $F(y;\alpha/2)$:

$$(K_{2/\alpha}^{0,1-\alpha}F)(y) = -\frac{2}{\alpha}\frac{\mathrm{d}}{\mathrm{d}y}(y^{-1}F(y;\alpha/2)).$$

In the case $1 < \alpha < 2$ we get the more complicated equation

$$\left(K_{2/\alpha}^{-1,2-\alpha}\left(1 - \frac{\alpha}{2}u\frac{\mathrm{d}}{\mathrm{d}u}\right)F\right)(y) = -\frac{2}{\alpha}\frac{\mathrm{d}}{\mathrm{d}y}(y^{-1}F(y;\alpha/2)).$$

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